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Characterization of Projection Lattices of Hilbert Spaces

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The classical lattices of projections of Hilbert spaces over the real, the complex, or the quaternion number field are characterized among the totality of irreducible, complete, orthomodular, atomic lattices satisfying the covering property. To this end, so-called paratopological lattices are introduced, i.e., lattices carrying a topology that renders the lattice operations restrictedly continuous.

INTRODUCTION

Birkhoff and von Neumann (1936) were the first to stress the lattice-theoretic aspect of the foundations of quantum mechanics. Mackey (1957) provided a set of axioms for quantum mechanics: among other things, he assumed that the poset of two-valued observables is isomorphic to the lattice of projections of some Hilbert space. Zierler (1961) examined what physically plausible assumptions on a lattice are sufficient to ensure the existence of an isomorphism onto the projection lattice of some separable real or complex Hilbert space. In order to force the reals $\mathbb{R}$ or the complex numbers $\mathbb{C}$ as coordinatizing fields of the examined class of lattices, he made use of Pontrjagin's characterization of $\mathbb{R}$ and $\mathbb{C}$ as the only locally compact, connected fields; he derived the premises of Pontrjagin's theorem roughly from the following two conditions: (1) There exists an arcwise connected ideal of finite height. (2) Any ideal of finite height is compact. (Zierler, 1961, pp. 1162ff). The underlying topology on the lattice is induced by a metric defined from the set of states on the lattice.

Varadarajan (1968) sketched the lattice-theoretic frame for further considerations on the axiomatization of quantum mechanics as follows: It should comprise the class $\mathscr{L}$ or irreducible, complete, orthomodular, atomic

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lattices enjoying the covering property (see also Beltrametti and Cassinelli, 1981; Maeda and Maeda, 1970; Piron, 1976). Since in the meantime Finkelstein *et al.* (1962) had pointed out the importance of quaternion quantum mechanics, the time had come to formulate the following problem (Varadarajan, 1968, p. 182):

Characterize the projection lattices of Hilbert spaces over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ (quaternion numbers) in the class $\mathscr{L}$ described above. (As usual I term these projection lattices classical.)

A solution of this problem in the separable case had already been given by Piron (1964) (cf. Varadarajan, 1968, p. 184), however, without clarifying why the three classical division rings come in.

This problem has since been treated successfully in various manners [cf. Beltrametti and Cassinelli, 1981, Chapter 21 for an overview] and a geometrically inspired solution will also be presented here.

In a subsequent paper I introduce the notion of a paratopological lattice, i.e., a lattice with top and bottom and a topology such that joining of skew (=meet-zero) elements and the dual operation are continuous mappings with open domains. Paratopological projective geometries have been studied in Szambien (1986); they correspond to the so-called topological projective spaces, which have a rather long history (cf. Doignon, 1971; Kolmogorov, 1932; Misfeld, 1968; Sörensen, 1969), and they are appropriate tools in solving the characterization problem. In Theorem 1, I single out some properties of the classical projection lattices. Property (b) therein sharpens Theorems (3.4) and (3.5) of Cirelli and Cotta-Ramusino (1973), while property (c) is equivalent to their Theorem (3.2). In the proof of Theorem 1 convergence structures [used in Cirelli and Cotta-Ramusino, (1973)] are replaced by topologies, i.e., systems of open sets. The properties exhibited in Theorem 1 are then used in Theorem 2 to characterize the classical projection lattices as certain lattices from $\mathscr{L}$ provided with an extrinsic topology enjoying some compatibility conditions. In the proof, results from topological geometry and again Pontrjagin's theorem are exploited. The compatibility conditions correspond to ($\mathscr{L}_1$), ($\mathscr{L}_2$), ($\mathscr{L}_4$), and ($\mathscr{L}_5$) in Cirelli and Cotta-Ramusino (1973) which there, however, are formulated with respect to the topology of states. ($\mathscr{L}_3$) is superfluous, because any nondiscrete, locally compact topological division ring is already first countable (cf., e.g., Bourbaki, 1972, VI.9.3). Theorem 2 corresponds to Theorem (1.1) of Cirelli *et al.* (1974) [or to Theorem (5.1) of Cirelli and Cotta-Ramusino (1973) in the separable case]. Central parts of the proof of Theorem (5.1) of Cirelli and Cotta-Ramusino (1973) can already be found in Pickert's (1955, p. 265) monograph on projective planes; even Schäfer, (1980) has not noticed this fact in his thesis. It finally should be pointed out that *a priori* the extrinsic topology from the first claim of

Theorem 2 is not connected with the topology of states; it remains an open problem whether this extrinsic topology must even coincide with the topology of states.

RESULTS

Let $\underset{\sim}{L}$ be a lattice with top 1 and bottom 0, $(\underset{\sim}{L}\times\underset{\sim}{L})_0$ be the set of pairs (a, b) from $\underset{\sim}{L}\times\underset{\sim}{L}$ with $a\wedge b=0$, $(\underset{\sim}{L}\times\underset{\sim}{L})_1$ be the set of pairs (a, b) with $a\vee b=1$, and τ be a topology on $\underset{\sim}{L}$; in the following "|" will always denote the restriction of mappings or topologies; $\downarrow a := \{b\in\underset{\sim}{L}\colon b\leq a\}$ denotes the principal ideal generated by $a\in\underset{\sim}{L}$. As a general reference for results from lattice theory I take Birkhoff (1960).

$(\underset{\sim}{L}, \tau)$ is called a *paratopological lattice* if, and only if, $\vee|(\underset{\sim}{L}\times\underset{\sim}{L})_0$ and $\wedge|(\underset{\sim}{L}\times\underset{\sim}{L})_1$ are continuous mappings with open domains. By an *orthocomplementation* of a lattice I mean a mapping $\perp:\underset{\sim}{L}\to\underset{\sim}{L}$ satisfying $a\wedge a^{\perp}=0$, $a\vee a^{\perp}=1$, $a^{\perp\perp}=a$, and $b^{\perp}\leq a^{\perp}$, if $a\leq b$, for all a, $b\in\underset{\sim}{L}$. I call *a orthogonal to b* $(a\perp b)$ iff $a^{\perp}\leq b$. Now let $\underset{\sim}{L}$ be a lattice with orthocomplementation. A state on $\underset{\sim}{L}$ is a mapping $s:\underset{\sim}{L}\to\mathbb{R}$ that fulfills $0\leq s(a)\leq 1$, $s(0)=0$, $s(1)=1$, $\sum s(a_i)=s(\sup\{a_i\colon i\in N\})$ for any sequence $(a_i)_{i\in\mathbb{N}}$ of pairwise orthogonal lattice elements. A state is called *pure* iff $s=tu+(1-t)v$ with states u, v, and $t\in\mathbb{R}$, $0<t<1$ always implies $u=v$. In the sequel, we consider the set Σ of all pure states on $\underset{\sim}{L}$, and σ, the coarsest topology on $\underset{\sim}{L}$ that renders continuous all elements of Σ; σ is called the topology of states on $\underset{\sim}{L}$.

Let $\underset{\sim}{H}$ be a Hilbert space with a dimension of at least 4 over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Let $\underset{\sim}{P}$ be the lattice of projections of $\underset{\sim}{H}$ provided with the respective topology of states. We then have:

Theorem 1. (a) The lattice $\underset{\sim}{P}$ is irreducible, complete, orthomodular, atomic, and has the covering property.

(b) The principal ideal generated by a projection of finite rank is a compact paratopological lattice with respect to the subspace structure inherited from $\underset{\sim}{P}$.

(c) The set of atoms of $\underset{\sim}{P}$ is connected.

Proof. Part (a) can be found in Maeda and Maeda (1970, Theorem 34.2).

Part (b). From Cirelli and Cotta-Ramusino (1973, Theorem 4.1) (cf. Varadarajan, 1968, Theorem 7.23), we know that σ equals the weak operator topology restricted to $\underset{\sim}{P}$. But on $\underset{\sim}{P}$ the weak and the strong operator topology coincide (cf., e.g., Kadison and Ringrose, 1983, p. 371).

Now let a be a projection of finite rank $\mathrm{rk}(a)$, and let $A := \mathrm{im}(a)$ denote its image space, ν the norm topology on $\underset{\sim}{H}$, $M\in\nu\,|\,A$ an open subset of A, $M^* := \{b \in \downarrow a\colon \mathrm{im}(b) \cap M \neq \emptyset\}$, and for $k \in \mathbb{Z}$, $0 \leq k \leq \mathrm{rk}(a)$, $S(k) := \{b \in \downarrow a\colon \mathrm{rk}(b) = k\}$. I then claim that

$$\underset{\sim}{B} := \{M^*\colon M \in \nu \mid A\} \cup \{S(k)\colon 0 \leq k \leq \mathrm{rk}(a)\}$$

is a subbase of the strong operator topology ω restricted to $\downarrow a$.

To prove this, I remark first of all that ω coincides with the point-open topology of mappings $\underset{\sim}{H} \to \underset{\sim}{H}$, when restricted to $\underset{\sim}{P}$. Now, M is clearly an element of the point-open topology restricted to $\downarrow a$; hence $M \in \omega \mid \downarrow a$.

Second, mapping $b \in \downarrow a$ to its trace (=rank) is continuous; thus, the sets $S(k)$ arise as clopen preimages, and $\underset{\sim}{B}$ is contained in $\omega \mid \downarrow a$. Since $\downarrow a$ is compact Hausdorff from Cirelli and Cotta-Ramusino (1973, Theorem 3.1) and since the topology generated by $\underset{\sim}{B}$ is Hausdorff by Szambien (1986, remark 2, p. 16) and Misfeld (1968, p. 261), it even equals $\omega \mid \downarrow a$. Hence the claim is proved.

Using the base $\underset{\sim}{B}$ of $\sigma \mid \downarrow a$, we infer from Szambien (1986, Theorem 3) that $\downarrow a$ is a paratopological lattice.

Part (c) is immediate, since the underlying coordinate field is connected. □

Theorem 2. Let $\underset{\sim}{L}$ be a lattice provided with a nondiscrete Hausdorff topology, such that conditions (a)-(c) of Theorem 1 are satisfied; then $\underset{\sim}{L}$ is orthoisomorphic to a classical lattice of projections. If, furthermore, the given topology coincides with the topology of states on $\underset{\sim}{L}$, then the orthoisomorphism is even a homeomorphism.

Proof. From condition (a) we get that $\underset{\sim}{L}$ is orthoisomorphic to the lattice of orthogonally closed subspaces of some vector space V over a certain division ring D. Orthogonality of subspaces in this vector space is given by some α-bilinear form on V, where α is an involutorial antiautomorphism of D; for details concerning this classical result of Birkhoff and von Neumann consult Maeda and Maeda (1970, Theorem 34.5). Now take $a \in \underset{\sim}{L}$ of finite height. Supposing (b), we infer that $\downarrow a$ is a paratopological projective geometry coordinatized by D. Since the given Hausdorff topology is nondiscrete, we can apply Pickert (1955, p. 265) to conclude that D is a topological field, which of course is locally compact (cf. also Szambien, 1986, proposition 3, remark 2). From Cirelli *et al.* (1974, Section 3) we know that α is continuous (see remark 4). As is well known, D may be either connected or totally disconnected. Now assuming (c), we infer using Salzmann (1955, p. 494) that D is connected. Then Pontrjagin's theorem yields that D equals one of $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Hence the continuity of α implies that α is the identity on $\mathbb{R}$, or complex, or quaternionic conjugation, respectively (see Varadarajan, 1968, pp. 181ff). Thus, V is a pre-Hilbert space. Since orthomodularity is assumed, the Amemiya-Araki Theorem (cf.

Maeda and Maeda, 1970, Theorem 34.9) can be applied to give that V is even complete. So the first part of the theorem is proved.

Now the orthoisomorphism noted above induces a bijection between the respective sets of pure states (cf. Cirelli *et al.*, 1974, *p.* 143). By definition of the topology of states and by the transitivity of the initial constructions used in defining these topologies, the orthoisomorphism is continuous and open. □

Remarks. (1) Hypothesis (c) can be weakened by requiring the connectedness of "affine lines" only (cf. Salzmann, 1955, p. 494; see also Cirelli and Cotta-Ramusino, 1973, p. 21). Instead of the Hausdorff property it suffices to suppose that the given topology is not indiscrete (cf. Pickert, 1955, p. 262).

(2) Following Schäfer, (c) can be replaced by the following interesting condition:

(d) The length of $\underline{L}$ is at least 5 and the fixed point set of the anti-automorphism $\alpha : D \to D$ of the coordinatizing division ring is contained in the center of D.

In Schäfer (1980) it is shown that (d) implies the connectedness of D (cf. also Wilbur, 1977, Theorem 5.8).

(3) The topology of states is Hausdorff iff the family of pure states is separating in the usual sense (cf. Cirelli and Cotta-Ramusino, 1973, p. 20).

(4) It is easily seen that Zierler's topology as defined in Zierler (1961, p. 1162) and the topology of states coincide on compact principal ideals of finite height. Therefore, the continuity of α is already proven by Zierler in his addendum (Zierler, 1966) to Zierler (1961).

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